

# LARGE WEIGHT DOES NOT YIELD AN IRREDUCIBLE BASE

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ABSTRACT. Answering a question of Juhász, Soukup and Szentmiklssy, we show that it is consistent that some first countable space of uncountable weight does not contain an uncountable subspace which has an irreducible base.

## 1. INTRODUCTION

For a topological space  $X$ ,  $w(X)$  is the minimal cardinality of a base for  $X$ ,  $\chi(p, X) = \min\{|u| : u \text{ is a neighbourhood base of } p\}$ , and  $\chi(X) = \sup\{\chi(p, X) : p \in X\}$ .

In [1] the following problem was investigated: What makes a space have weight larger than its character? The notion of *irreducible base* was introduced, and it was proved [1, Lemma 2.6] that *if a topological space  $X$  has an irreducible base then  $w(X) = |X| \cdot \chi(X)$* . The following question was formulated:

**Problem 1.** *Does every first countable space of uncountable weight contain an uncountable subspace which has an irreducible base?*

We show that the answer is consistently NO. We thank Lajos Soukup for actually writing the paper.

**Definition 1.1.** Let  $X$  be a topological space. A base  $\mathcal{U}$  of  $X$  is called *irreducible* if it has an *irreducible decomposition*  $\mathcal{U} = \bigcup\{\mathcal{U}_x : x \in X\}$ , i.e, (i) and (ii) below hold:

- (i)  $\mathcal{U}_x$  is a neighbourhood base of  $x$  in  $X$  for each  $x \in X$ .
- (ii) for each  $x \in X$  the family  $\mathcal{U}_x^- = \bigcup_{y \neq x} \mathcal{U}_y$  is not a base of  $X$ .

**Theorem 1.2.** *There is a c.c.c poset  $P = \langle P, \leq \rangle$  of size  $\omega_1$  such that in  $V^P$  there is a first countable space  $X = \langle \omega_1, \tau \rangle$  of uncountable weight which does not contain an uncountable subspace which has an irreducible base.*

*Proof.* The elements of the poset  $P$  will be finite “approximations” of a base  $\{U(\alpha, n) : \alpha < \omega_1, n < \omega\}$  of  $X$ .

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We define the poset  $P = \langle P, \leq \rangle$  as follows. The underlying set of  $P$  consists of the triples  $\langle A, n, U \rangle$  satisfying (P1)–(P3) below:

- (P1)  $A \in [\omega_1]^{<\omega}$ ,  $n \in \omega$  and  $U$  is a function,  $U : A \times n \rightarrow \mathcal{P}(A)$ ,
- (P2)  $\alpha \in U(\alpha, i) \subset U(\alpha, i-1)$  for each  $\alpha \in A$  and  $i < n$ ,
- (P3) If  $\beta \in U(\alpha, i) \subset U(\beta, 0)$  for some  $i < n$ , then  $\beta \leq \alpha$ .

For  $p \in P$  write  $p = \langle A_p, n_p, U_p \rangle$ . Let us remark that property (P3) will guarantee that  $w(X) = \omega_1$ .

Define the order  $\leq$  on  $P$  as follows. For  $p, q \in P$  we put  $q \leq p$  if

- (a)  $A_p \subset A_q$ ,
- (b)  $n_p \leq n_q$ ,
- (c)  $U_p(\alpha, i) = U_q(\alpha, i) \cap A_p$  for each  $\langle \alpha, i \rangle \in A_p \times n_p$ ,
- (d) for each  $\langle \alpha, i \rangle, \langle \beta, j \rangle \in A_p \times n_p$ ,
  - if  $U_p(\alpha, i) \cap U_p(\beta, j) = \emptyset$  then  $U_q(\alpha, i) \cap U_q(\beta, j) = \emptyset$ , (d1)
  - if  $U_p(\alpha, i) \subset U_p(\beta, j)$  then  $U_q(\alpha, i) \subset U_q(\beta, j)$ . (d2)

We say that the conditions  $p_0 = \langle A_0, n_0, U_0 \rangle$  and  $p_1 = \langle A_1, n_1, U_1 \rangle$  are *twins* iff  $n_0 = n_1$ ,  $|A_0| = |A_1|$  and denoting by  $\sigma$  the unique  $<_{\text{On}}$ -preserving bijection between  $A_0$  and  $A_1$  we have

- (I1)  $\sigma \upharpoonright A_0 \cap A_1 = \text{id}_{A_0 \cap A_1}$ ,
- (I2)  $\sigma$  is an isomorphism between  $p_0$  and  $p_1$ , i.e. for each  $\alpha \in A_0$  and  $i < n_0$  we have  $U_1(\sigma(\alpha), i) = \sigma''U_0(\alpha, i)$ .

We say that  $\sigma$  is the *twin function* between  $p_0$  and  $p_1$ . Define the *smashing function*  $\underline{\sigma}$  of  $p_0$  and  $p_1$  as follows:  $\underline{\sigma} = \sigma^{-1} \cup \text{id}_{A_0}$ . The function  $\sigma^*$  defined by the formula  $\sigma^* = \sigma \cup \sigma^{-1}$  is called the *exchange function* of  $p_0$  and  $p_1$ .

The burden of the proof is to verify the next lemma.

**Amalgamation Lemma 1.3.** *Assume that  $p_0 = \langle A_0, n_0, U_0 \rangle$  and  $p_1 = \langle A_1, n_1, U_1 \rangle$  are twins,  $A_0 \cap A_1 < A_0 \setminus A_1 < A_1 \setminus A_0$ ,  $\xi_0 \in A_0 \setminus A_1$ ,  $\xi_1 = \sigma(\xi_0)$ , where  $\sigma$  is the twin function between  $p_0$  and  $p_1$ , and let  $k < m < n_0$ . Then  $p_0$  and  $p_1$  have a common extension  $p = \langle A, n, U \rangle$  in  $P$  such that*

$$\xi_0 \in U(\xi_1, m) \subset U(\xi_1, k) \subset U(\xi_0, k). \quad (*)$$

*Proof.* Write  $n = n_0 = n_1$ ,  $D = A_0 \cap A_1$  and  $A^* = A_0 \cup A_1$ . Unfortunately we can not assume that  $A = A^*$  because in this case we can not guarantee (P3) for  $p$ . So we need to add further elements to  $A^*$  to get a large enough  $A$  as follows. Choose a set  $B \subset \omega_1 \setminus A^*$  of cardinality  $|A^* \times n|$  and fix a bijection  $\rho$  between  $A^* \times n$  and  $B$ . We will take  $A = A^* \cup B$ . To simplify the notation we will write  $\langle \alpha, i \rangle$  for  $\rho(\alpha, i)$ , for all  $\alpha \in A^*$  and  $i < n$ , i.e. we identify the elements of  $B$  and of  $A^* \times n$ .

The idea of the proof is the following: for each  $\langle \alpha, i \rangle \in A^* \times n$  we put the element  $\langle \alpha, i \rangle$  into  $U(\alpha, i)$ . On the other hand, we try to keep  $U(\alpha, i)$  small, so we put  $\langle \beta, j \rangle$  into  $U(\alpha, i)$  if and only if we can “derive” from the property (d2) that  $U(\beta, j) \subset U(\alpha, i)$  should hold in

any condition  $p = \langle A, n, U \rangle$  which is a common extension of  $p_0$  and  $p_1$  and which satisfies  $(*)$ .

The condition  $p$  will be constructed in two steps. First we construct a condition  $p' = \langle A, n, U' \rangle$  extending both  $p_0$  and  $p_1$ . This  $p'$  can be considered as the minimal amalgamation of  $p_0$  and  $p_1$ . Then, in the second step, we carry out small modifications on the function  $U'$ , namely we increase its value on certain places to guarantee  $(*)$ .

Now we carry out our construction. For  $\varepsilon < 2$  and  $\langle \beta, j \rangle \in A_\varepsilon \times n$  let

$$V_\varepsilon(\beta, j) = \{ \langle \alpha, i \rangle \in A_\varepsilon \times n : U_\varepsilon(\alpha, i) \subset U_\varepsilon(\beta, j) \} \quad (1)$$

and

$$W_\varepsilon(\beta, j) = \{ \langle \alpha, i \rangle \in A_{1-\varepsilon} \times n : \exists \langle \gamma, l \rangle \in D \times n \\ U_{1-\varepsilon}(\alpha, i) \subset U_{1-\varepsilon}(\gamma, l) \wedge U_\varepsilon(\gamma, l) \subset U_\varepsilon(\beta, j) \} \quad (2)$$

If we want to define  $p'$  in such a way that  $p' \leq p_0, p_1$ , then (d2) implies that  $U'(\alpha, i) \subset U'(\beta, j)$  should hold whenever  $\langle \alpha, i \rangle \in V(\beta, j) \cup W(\beta, j)$ .

Now we are ready to define the function  $U'$ . For  $\varepsilon < 2$ ,  $\beta \in A_\varepsilon$  and  $j < n$  let

$$U'(\beta, j) = U_\varepsilon(\beta, j) \cup U_{1-\varepsilon}(\sigma^*(\beta), j) \cup V_\varepsilon(\beta, j) \cup W_\varepsilon(\beta, j). \quad (3)$$

For  $\langle \alpha, i \rangle \in A^* \times n$  and  $j < n$  let

$$U'(\langle \alpha, i \rangle, j) = \{ \langle \alpha, i \rangle \}. \quad (4)$$

Let us remark that  $U'(\delta, j)$  is well-defined even for  $\delta \in A_0 \cap A_1$ . Indeed, in this case  $\sigma^*(\delta) = \delta$  and  $V_\varepsilon(\delta, j) = W_{1-\varepsilon}(\delta, j)$ , and so

$$U'(\delta, j) = U_0(\delta, j) \cup U_1(\delta, j) \cup V_0(\delta, j) \cup V_1(\delta, j).$$

Now put

$$p' = \langle A, n, U' \rangle.$$

**Claim 1.4.** *If  $\alpha \in U'(\beta, j)$  then  $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$ .*

Indeed, if  $\beta \in A_\varepsilon$  then  $U'(\beta, j) \cap A^* = U_\varepsilon(\beta, j) \cup U_{1-\varepsilon}(\sigma^*(\beta), j)$ .

**Claim 1.5.** *If  $\langle \alpha, i \rangle \in U'(\beta, j)$  then  $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$ .*

*Proof of the Claim.* Assume that  $\beta \in A_\varepsilon$ . If  $\langle \alpha, i \rangle \in V_\varepsilon(\beta, j)$  then  $\alpha \in U_\varepsilon(\alpha, i) \subset U_\varepsilon(\beta, j)$  and  $U_\varepsilon(\beta, j) \subset U'(\beta, j)$ . So we have  $\alpha \in U'(\beta, j)$  which implies  $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$  by Claim 1.4.

If  $\langle \alpha, i \rangle \in W_{1-\varepsilon}(\beta, j)$  then for some  $\langle \gamma, l \rangle \in D \times n$  we have  $U_{1-\varepsilon}(\alpha, i) \subset U_{1-\varepsilon}(\gamma, l) \wedge U_\varepsilon(\gamma, l) \subset U_\varepsilon(\beta, j)$ . Thus  $\alpha \in U_\varepsilon(\beta, j) \subset U'(\beta, j)$ , which implies  $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$  by Claim 1.4.  $\square_{1.5}$

**Claim 1.6.**  $p' \in P$ .

*Proof of the claim 1.6.* (P1) and (P2) clearly hold, so we need to check only (P3).

Assume on the contrary that (P3) fails for  $p'$ . Since  $U'(\langle \nu, s \rangle, j) = \{ \langle \nu, s \rangle \}$  by (4) for each  $\langle \nu, s \rangle \in B$  and  $j < n$ , we can assume that some

$\alpha < \beta \in A^*$  and  $i < n$  witness that (P3) fails, i.e.  $\beta \in U'(\alpha, i) \subset U'(\beta, 0)$ . Then  $\underline{\sigma}(\beta) \in U_0(\underline{\sigma}(\alpha), i) \subset U'(\underline{\sigma}(\beta), 0)$  by Claim 1.4. Since  $p_0$  satisfies (P3) it follows that  $\underline{\sigma}(\beta) \leq \underline{\sigma}(\alpha)$ , and so  $\alpha \in A_0 \setminus A_1$  and  $\beta \in A_1 \setminus A_0$ . Consider the element  $u = \langle \alpha, i \rangle \in A \setminus A^*$ . Then  $u \in U'(\alpha, i)$  and so  $u \in U'(\beta, 0)$  as well. By the definition of  $U'(\beta, 0)$  this means that  $\langle \alpha, i \rangle \in W_1(\beta, 0)$ , that is, there is  $\langle \gamma, l \rangle \in D \times n$  such that  $U_0(\alpha, i) \subset U_0(\gamma, l)$  and  $U_1(\gamma, l) \subset U_1(\beta, j)$ . Thus

$$\underline{\sigma}(\beta) \in U_0(\alpha, i) \subset U_0(\gamma, l) \subset U_0(\underline{\sigma}(\beta), 0) \quad (5)$$

by Claim 1.4. Thus  $\underline{\sigma}(\beta) \in U_0(\gamma, l) \subset U_0(\underline{\sigma}(\beta), 0)$  and so  $\underline{\sigma}(\beta) \leq \gamma$  because  $p_0$  satisfies (P3). But this is a contradiction because  $\gamma \in D = A_0 \cap A_1$ ,  $\underline{\sigma}(\beta) \in A_0 \setminus A_1$  and we assumed that  $(A_0 \cap A_1) < (A_0 \setminus A_1)$ .  $\square_{1.6}$

**Claim 1.7.**  $p' \leq p_0, p_1$ .

*Proof of claim 1.7.* Conditions (a) and (b) are clear.

To check (c) assume that  $\alpha \in A_\varepsilon$  and  $i \in n$ . By (3),

$$\begin{aligned} U'(\alpha, i) \cap A_\varepsilon &= (U_\varepsilon(\alpha, i) \cup U_{1-\varepsilon}(\alpha, i)) \cap A_\varepsilon = \\ &= U_\varepsilon(\alpha, i) \cup (U_{1-\varepsilon}(\alpha, i) \cap A_\varepsilon) = U_\varepsilon(\alpha, i) \end{aligned}$$

because  $U_{1-\varepsilon}(\alpha, i) = \sigma^*[U_\varepsilon(\alpha, i)]$ .

To check (d1) assume that  $\beta, \gamma \in A_\varepsilon$  and  $j, k < n$  such that  $U'(\beta, j) \cap U'(\gamma, k) \neq \emptyset$ . Fix  $x \in U'(\beta, j) \cap U'(\gamma, k)$ . Then

$$\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j) \cap U_0(\underline{\sigma}(\gamma), k)$$

by Claim 1.4 if  $x = \alpha \in A^*$ , and by Claim 1.5 if  $x = \langle \alpha, i \rangle \in A \setminus A^*$ .

If  $\varepsilon = 0$  then  $\underline{\sigma}(\beta) = \beta$  and  $\underline{\sigma}(\gamma) = \gamma$ , so  $\underline{\sigma}(\alpha) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$ .

If  $\varepsilon = 1$  then  $\underline{\sigma}(\beta) = \sigma^*(\beta)$  and  $\underline{\sigma}(\gamma) = \sigma^*(\gamma)$ , and so  $\sigma^*(\underline{\sigma}(\alpha)) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$ .

Finally to check (d2) assume that  $\beta, \gamma \in A_\varepsilon$  and  $j, k < n$  such that  $U_\varepsilon(\beta, j) \subset U_\varepsilon(\gamma, k)$ . Then clearly

$$U_{1-\varepsilon}(\beta, j) = \sigma^*[U_\varepsilon(\beta, j)] \subset \sigma^*[U_\varepsilon(\gamma, k)] = U_{1-\varepsilon}(\gamma, k),$$

moreover  $V_\varepsilon(\beta, j) \subset V_\varepsilon(\gamma, k)$  by (1), and  $W_\varepsilon(\beta, j) \subset W_\varepsilon(\gamma, k)$  by (2), and so  $U'(\beta, j) \subset U'(\gamma, k)$  by (4).  $\square_{1.7}$

Now carry out the promised modification of  $U'$  to obtain  $U$  as follows. If  $z \in A$  and  $j < n$  let

$$U(z, j) = \begin{cases} U'(z, j) \cup U'(\xi_1, k) & \text{if } U_0(\xi_0, k) \subset U_0(z, j), \\ U'(z, j) & \text{otherwise.} \end{cases}$$

Put

$$p = \langle A, n, U \rangle.$$

If  $U_0(\xi_0, k) \subset U_0(z, j)$  then  $U_1(\xi_1, k) \subset U_1(\sigma^*(z), j) \subset U'(z, j)$  and  $W_1(\xi_1, k) \subset V_0(\xi_0, k) \subset U'(z, j)$ . So

$$U(z, j) \setminus U'(z, j) \subset V_1(\xi, k). \quad (6)$$

Moreover

$$U(z, j) = \begin{cases} U'(z, j) \cup V_1(\xi_1, k) & \text{if } U_0(\xi_0, k) \subset U_0(z, j), \\ U'(z, j) & \text{otherwise.} \end{cases} \quad (7)$$

**Claim 1.8.** *If  $\langle \alpha, i \rangle \in U(\beta, j)$  then  $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$ .*

Indeed, if  $\langle \alpha, i \rangle \in U(\beta, j)$  then  $\langle \alpha, i \rangle \in U'(\beta, j)$  or  $\langle \alpha, i \rangle \in U'(\sigma^*(\beta), j)$ , and now apply Claim 1.5.

**Claim 1.9.**  $p \in P$ .

*Proof of claim 1.9.* (P1) and (P2) clearly hold, so we need to check (P3) only.

Assume on the contrary that (P3) fails for  $p$ . Since  $U(\langle \nu, s \rangle, j) = \{\langle \nu, s \rangle\}$  for each  $\langle \nu, s \rangle \in A \setminus A^*$  and  $j < n$  we can assume that there are  $\alpha < \beta \in A^*$  and  $i < n$  witness that (P3) fails, i.e.

$$\beta \in U(\alpha, i) \subset U(\beta, 0). \quad (8)$$

Then  $\underline{\sigma}(\beta) \in U_0(\underline{\sigma}(\alpha), i) \subset U(\underline{\sigma}(\beta), 0)$ . But  $p_0$  satisfies (P3) so  $\underline{\sigma}(\beta) \leq \underline{\sigma}(\alpha)$ , and so  $\alpha \in A_0 \setminus A_1$  and  $\beta \in A_1 \setminus A_0$ . Thus  $U_0(\beta, j)$  is undefined, and so

$$U'(\beta, 0) = U(\beta, 0) \text{ and } U(\alpha, i) \setminus U'(\alpha, i) \subset A \setminus A^*. \quad (9)$$

by (7). So (8) yields

$$\beta \in U'(\alpha, i) \subset U'(\beta, 0),$$

However this is a contradiction because  $p'$  satisfies (P3).  $\square_{1.9}$

**Claim 1.10.**  $p \leq p_0, p_1$ .

*Proof.* (a) and (b) are trivial. (c) also holds because  $p' \leq p_\varepsilon$  and  $(U(\alpha, i) \setminus U'(\alpha, i)) \cap A_\varepsilon = \emptyset$  by (6.)

To check (d1) assume that  $\beta, \gamma \in A_\varepsilon$  and  $j, k < n$  such that  $U(\beta, j) \cap U(\gamma, k) \neq \emptyset$ . Pick  $x \in U(\beta, j) \cap U(\gamma, k)$ . Then

$$\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j) \cap U_0(\underline{\sigma}(\gamma), k)$$

by Claim 1.4 if  $x = \alpha \in A^*$ , and by Claim 1.8 if  $x = \langle \alpha, i \rangle \in A \setminus A^*$ .

If  $\varepsilon = 0$  then  $\underline{\sigma}(\beta) = \beta$  and  $\underline{\sigma}(\gamma) = \gamma$ , so  $\underline{\sigma}(\alpha) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$ .

If  $\varepsilon = 1$  then  $\underline{\sigma}(\beta) = \sigma^*(\beta)$  and  $\underline{\sigma}(\gamma) = \sigma^*(\gamma)$ , and so  $\sigma^*(\underline{\sigma}(\alpha)) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$ .

Finally to check (d2) assume that  $\beta, \gamma \in A_\varepsilon$  and  $i, j < n$  such that  $U_\varepsilon(\beta, i) \subset U_\varepsilon(\gamma, j)$ . Since  $p' \leq p_\varepsilon$  we have  $U'(\beta, i) \subset U'(\gamma, j)$ . If  $U(\beta, i) = U'(\beta, i)$ , we are done. So we can assume that  $U(\beta, i) = U'(\beta, i) \cup V(\xi_1, k)$ . Then  $\varepsilon = 0$  and  $U_0(\xi_0, k) \subset U_0(\beta, i)$ . But then  $U_0(\xi_0, k) \subset U_0(\gamma, j)$  and so  $U(\gamma, j) = U'(\gamma, j) \cup V(\xi_1, k)$ , and so  $U(\beta, i) \subset U(\gamma, j)$ .  $\square_{1.10}$

Since  $p$  satisfies (\*), the amalgamation lemma is proved.  $\square_{1.3}$

Using the amalgamation lemma it is easy to complete the proof of the theorem.

By standard  $\Delta$ -system argument, any uncountable set of conditions contains two elements,  $p_0$  and  $p_1$ , which are twins. So, by Lemma 1.3, they have a common extension  $p$ . So  $P$  satisfies c.c.c.

If  $\mathcal{G}$  is a generic filter, for  $\alpha < \omega_1$  and  $i < \omega$  put

$$U(\alpha, i) = \cup \{U_p(\alpha, i) : p \in \mathcal{G}, \alpha \in A_p, i < n_p\}, \quad (10)$$

and let  $\mathcal{U}_\alpha = \{U(\alpha, i) : i < \omega\}$  be the base of the point  $\alpha$  in  $X = \langle \omega_1, \tau \rangle$ .

By (P3), a countable subfamily of  $\{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$  is not a base of  $X$ . So  $w(X) = \omega_1$ .

Finally we show that  $X$  does not contain an uncountable subspace which has an irreducible base.

Assume on the contrary that

$$\begin{aligned} r \Vdash \text{the subspace } \dot{Y} = \{\dot{y}_\xi : \xi < \omega_1\} \text{ has an irreducible base } \mathcal{B}, \\ \text{and } \{\dot{\mathcal{B}}_{\dot{y}_\xi} : \xi < \omega_1\} \text{ is an irreducible decomposition of } \dot{\mathcal{B}}. \end{aligned}$$

We can assume that  $r \Vdash \dot{y}_\xi \geq \check{\xi}$ .

For each  $\xi < \omega_1$  pick a condition  $r_\xi$  and  $k_\xi \in \omega$  such that

$$r_\xi \Vdash \text{“if } V \in \mathcal{B} \text{ with } \dot{y}_\xi \in V \subset U(\dot{y}_\xi, \check{k}_\xi) \text{ then } V \in \mathcal{B}_{\dot{y}_\xi}\text{”}. \quad (11)$$

For each  $\xi < \omega_1$  pick a condition  $p_\xi \leq r_\xi$ , an ordinal  $\alpha_\xi \geq \xi$ , a name  $\dot{V}_\xi$  and a natural number  $m_\xi < \omega$  such that  $\alpha_\xi \in A_{p_\xi}$  and

$$p_\xi \Vdash \dot{y}_\xi = \check{\alpha}_\xi, \dot{V}_\xi \in \dot{\mathcal{B}}_{\alpha_\xi} \text{ and } U(\check{\alpha}_\xi, \check{m}_\xi) \subset \dot{V}_\xi \subset U(\check{\alpha}_\xi, \check{k}_\xi). \quad (12)$$

By standard argument find  $I \in [\omega_1]^{\omega_1}$  such that

- (i)  $m_\xi = m$  and  $k_\xi = k$  for each  $\xi \in I$ ,
- (ii) the sequence  $\{\alpha_\xi : \xi \in I\}$  is strictly increasing,
- (iii) the conditions  $\{p_\xi : \xi \in I\}$  are pairwise twins,
- (iv)  $\sigma_{\xi, \eta}(\alpha_\xi) = \alpha_\eta$  for  $\{\xi, \eta\} \in [I]^2$ , where  $\sigma_{\xi, \eta}$  is the twin function.

Pick  $\xi < \eta$  from  $I$ . By the Amalgamation Lemma there is a common extension  $p$  of  $p_\xi$  and  $p_\eta$  such that

$$\alpha_\xi \in U_p(\alpha_\eta, m) \wedge U_p(\alpha_\eta, k) \subset U_p(\alpha_\xi, k). \quad (13)$$

Then, by (d2),

$$p \Vdash \check{\alpha}_\xi \in U(\check{\alpha}_\eta, \check{m}) \wedge U(\check{\alpha}_\eta, \check{k}) \subset U(\check{\alpha}_\xi, \check{k}). \quad (14)$$

Then, by (12),

$$p \Vdash \dot{V}_\eta \in \mathcal{B}_{\alpha_\eta} \text{ and } \check{\alpha}_\xi \in U(\check{\alpha}_\eta, \check{m}) \subset \dot{V}_\eta \subset U(\check{\alpha}_\eta, \check{k}) \subset U(\check{\alpha}_\xi, \check{k}), \quad (15)$$

which contradicts (11).

This completes the proof of the Theorem.  $\square$

## REFERENCES

- [1] Juhsz, I.; Soukup, L.; Szentmiklssy, Z. *What makes a space have large weight?* Topology Appl. 57 (1994), no. 2-3, 271–285.

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